International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 1, pp: 1130-1135 www.ijaem.net ISSN: 2395-5252

Applicability of Covariant Formulation of Second Law of Motion

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Date of Submission: 30-6-2020 Date of Acceptance: 15-07-2020

ABSTRACT
The covariant formulation of Newton's second law using Energy-Momentum four-vector is derived. The application of this formulated covariant law successfully explains the observation of some pseudo-forces in nature. This verifies the mass-energy conversion formula for a hypothetical process of material particle production from a photon. This formulation also suggests a relationship between the electromagnetic potentials and the electromagnetic fields.

Keywords- Covariance, Four- vectors, Lorentz Invariance, Minkowski Space time, Relativity

I. INTRODUCTION

The covariant formulation refers to ways of writing the laws of physics in a form that is manifestly invariant under Lorentz transformation.
The invariance under Lorentz transformation expresses the proposition that the laws of physics are same for the observers in different frame of reference in relative motion to each other [1].That means the laws of Physics take the same form in co-ordinate system of any arbitrary frame of reference. Covariant formulation provides a way to translate physical variables from one frame of reference to another [2]. Principle of Covariance requires the formulation of physical laws using only those variables the measurement of which the observer in different frames of reference can unambiguously correlate. These are the quantities which have spatial as well as temporal components such as four-co-ordinates, four-vectors, four-momentum andfour-field [3].

We know Newton's Second law of motion tells us "The rate of change of momentum is proportional to the applied force and takes place in its direction". It has two quantities – the applied force and the momentum. The applied force is also the gradient of potential energy. Therefore we have two quantities – the momentum and the energy, which are part of a four-vector called four-momentum or Energy-Momentum four-vector. Therefore the covariant formulation of the second law, as we shall discuss below, can be accomplished using Energy-Momentum four-vector

II. DISCUSSION

Let us assume that a particle P at a point defined by co-ordinates (x, y, z)is moving with a momentum$\mathbf{p}$ after being applied by a force $\mathbf{F}$.Newton's Second law of motion, as stated above, tells us that rate of change of momentum is proportional to the applied force and takes place in its direction. Mathematically we can write it as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad (1)$$

As we know, the units of force are so defined to make the constant of proportionality as unity. The components of force $\mathbf{F}$ and momentum $\mathbf{p}$ are given by the familiar expressions $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$where F_x, F_y, F_z are components of the force,and $\mathbf{p} = p_x\hat{\mathbf{i}} + p_y\hat{\mathbf{j}} + p_z\hat{\mathbf{k}}$ wherep_x, p_y, p_z are components of the momentum. Then from above equation, x-component of the force will be given by

$$F_x = \frac{\partial p_x}{\partial t} \qquad (2)$$

Of course, this force can be a field, therefore possessing unique value at each point. Extending it to four dimensional concept let $\boldsymbol{E}$be the event of thisforce being applied to particle P attime say t. We have the co-ordinates of event $\boldsymbol{E}$ are, then x, y, z and t.The time co-ordinate is taken as 'ict' where c is the velocity of Light. Let the particle has a potential energy V due to this field. Then, we know force is a gradient of the potential energy or wecan write

$$\mathbf{F} = -\nabla V \qquad (3)$$

where $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$ is the gradient operator.

The x-component of Eq (3)is given by

$$F_x = -\frac{\partial V}{\partial x} \quad (4)$$

Comparing the two equations we get

$$-\frac{\partial V}{\partial x} = \frac{\partial p_x}{\partial t} \quad (5)$$

This result, however is not covariant in the sense **p** is a three vector. The corresponding four -vector is $\vec{p} = \left(\frac{iE}{c}, p_x, p_y, p_z\right)$, E being the total energy of the particle, c being the velocity of light.The four coordinates are given by $x = (ict, x, y, z)$. The only modification we can do to make above equation covariant is replacing V by E

$$-\frac{\partial E}{\partial x} = \frac{\partial p_x}{\partial t} \quad (6)$$

Dividing both sides by ic, We get

$$-\left(\frac{1}{ic}\right)\frac{\partial E}{\partial x} = \frac{\partial p_x}{\partial (ict)}$$

We can write as

$$\frac{\partial\left(\frac{iE}{c}\right)}{\partial x} = \frac{\partial p_x}{\partial (ict)} \quad (7)$$

Replacing coordinates by $x = (ict, x, y, z)$ by $x^\mu = (x^0, x^1, x^2, x^3)$ and $\vec{p} = \left(\frac{iE}{c}, p_x, p_y, p_z\right)$ by $\vec{p} = (p^0, p^1, p^2, p^3)$, we can write above equation as

$$\frac{\partial p^0}{\partial x^1} = \frac{\partial p^1}{\partial x^0}$$

Rearranging, we have

$$\frac{\partial p^1}{\partial x^0} - \frac{\partial p^0}{\partial x^1} = 0$$

We can write

$$\frac{\partial p^\nu}{\partial x^\mu} - \frac{\partial p^\mu}{\partial x^\nu} = 0 \quad (8)$$

for $\mu = 0$ and $\nu = 1$.

Further using tensor notations ∂_μ for $\frac{\partial}{\partial x^\mu}$ etc., Equation (8) becomes

$$\partial_\mu p^\nu - \partial_\nu p^\mu = 0 \quad (9)$$

for $\mu = 0$ and $\nu = 1$.

Since we are using co-ordinates are $x = (ict, x, y, z)$, $\vec{p} = \left(\frac{iE}{c}, p_x, p_y, p_z\right)$. We have metric $g_{\mu\nu} = g^{\mu\nu} = (1, 1, 1, 1) = \delta^\mu_\nu$.Therefore Equation (9) can be equivalently written as

$$\partial^\mu p^\nu - \partial^\nu p^\mu = 0 \quad (10)$$

and

$$\partial_\mu p_\nu - \partial_\nu p_\mu = 0 \quad (11)$$

for $\mu = 0$ and $\nu = 1$.

We see that Equation (11) is $\mu\nu$ component of an anti-symmetric tensor $\partial \wedge p$ for $\mu = 0$ and $\nu = 1$. Let us generalize this result assuming it true for all μ and ν,we can write the covariant form substitute for Newton's Second Law as

$$\boldsymbol{\partial \wedge p = 0} \quad (12)$$

The tensor $\boldsymbol{\partial \wedge p}$ has six components given by $(\boldsymbol{\partial \wedge p})_{\boldsymbol{\mu\nu}} = \partial_\mu p_\nu - \partial_\nu p_\mu$ for $\mu, \nu =$ 0,1,2,3.Therefore assigning different values to $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, we get six relations between four components of Energy-Momentum four-vector. Choosing different values of μ and ν, we get the following six relations

$$-\frac{\partial E}{\partial x} = \frac{\partial p_x}{\partial t} \quad \text{(i)}$$

$$-\frac{\partial E}{\partial y} = \frac{\partial p_y}{\partial t} \quad \text{(ii)}$$

$$-\frac{\partial E}{\partial z} = \frac{\partial p_z}{\partial t} \quad \text{(iii)}$$

$$\frac{\partial p_y}{\partial x} = \frac{\partial p_x}{\partial y} \quad \text{(iv)}$$

$$\frac{\partial p_z}{\partial x} = \frac{\partial p_x}{\partial z} \quad \text{(v)}$$

$$\frac{\partial p_z}{\partial y} = \frac{\partial p_y}{\partial z} \quad \text{(vi)}$$

(13)

The first three relations are between spatial variation of energy and the temporal rate of change of three-momentum. The last three relations just correlate the spatial variations of the components of the three- momentum. These relations prove the fact that whenever there is a spatial variation in Energy; a force appears in the direction of the decrease in energy. This force we usually call as pseudo- force. The examples of these forces are upward force in a falling lift or frame of reference, centrifugal force on a particle doing circular motion etc.

Following is the discussion on illustrative examples of these results.

III. ILLUSTRATIVE EXAMPLES

The illustrative examples chosen here have kinetic energy variations which are the prominent energy variations in these cases but Tensor Equation (12) is equally and actually valid for the total energy variations only.

A. Pseudo force in a freely falling frame

Let us consider an object of mass m in a frame of reference falling vertically downward along z-axis with acceleration $-\mathbf{a_z}$ (negative sign to show that the acceleration is along negative z axis) as shown in Figure.1.We get from (13(iii))

$$\frac{\partial E}{\partial z} + \frac{\partial p_z}{\partial t} = 0 \qquad (14)$$

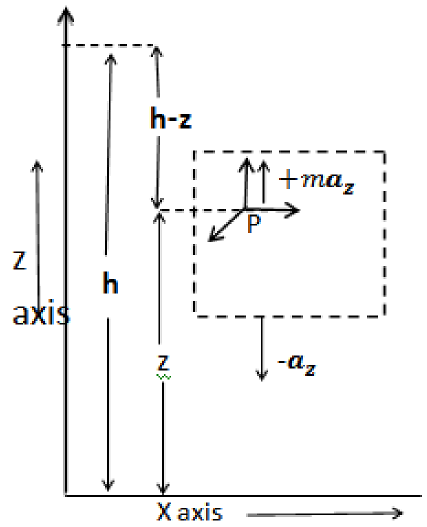

Figure 1: Frame at particle P in a container moving downwards with acceleration $\boldsymbol{a_z}$. The particle experiences a force $m\boldsymbol{a_z}$ upwards.

Let the frame falls from a height h. We have, change in energy at height z is equal to gain in the kinetic energy $\frac{1}{2}m\dot{z}^2$. But we have from elementary kinematics $0 - \dot{z}^2 = -2a_z(h - z)$. Therefore the increase in the kinetic energy is $ma_z(h - z)$.We get from Equation (14)

$$\frac{\partial p_z}{\partial t} = -\frac{\partial E}{\partial z} = -\frac{\partial(ma_z(h - z))}{\partial z} = ma_z \qquad (15)$$

Therefore the object experiences a force ma_z along positive z direction.

B. Centrifugal Force

Consider a particle of mass m moving in a circle of radius r with an angular velocity ω as shown in Figure.2. The velocity of the particle is $v = \omega r$. The kinetic energy, which is the only variable part of the total energy of the particle under non-relativistic conditions, is given by

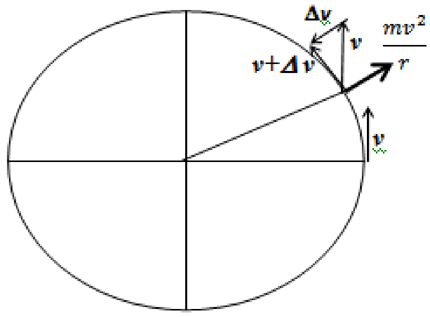

Figure 2: Particle rotating in circular path with velocity v experiences an outward force of $\frac{mv^2}{r}$.

$$E = \frac{1}{2}mv^2$$

$$\frac{\partial E}{\partial r} = mv\frac{\partial v}{\partial r} \qquad (16)$$

Therefore

The change in velocity is $\Delta v = \omega\Delta r$ and is along $-\Delta r$ i.e. points towards the center of the circle. Therefore $\Delta v = -\omega\Delta r$. For small variations, this becomes

$$\frac{\partial v}{\partial r} = \lim_{\Delta r \to 0}\frac{\Delta v}{\Delta r} = -\omega$$

Therefore from Equation (16), we have

$$\frac{\partial E}{\partial r} = -mv\omega \qquad (17)$$

We have from Equation (13(i))

$$\frac{\partial E}{\partial x} + \frac{\partial p_x}{\partial t} = 0 \qquad (18)$$

Let the particle is at an infinitely small displacement from the x-axis, we can safely replace x by r, and we get

$$\frac{\partial E}{\partial r} + \frac{\partial p_r}{\partial t} = 0$$

Therefore from Equation (17), we get

$$\frac{\partial p_r}{\partial t} = -\frac{\partial E}{\partial r} = mv\omega \qquad (19)$$

Therefore

$$\frac{\partial p_r}{\partial t} = \frac{mv^2}{r} \text{ as } \omega = \frac{v}{r} \qquad (20)$$

which is obviously an outward force acting along positive r direction, and is known as centrifugal force.

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 1, pp: 1130-1135 www.ijaem.net ISSN: 2395-5252

C. Conversion of Energy into mass

Let a photon of energy $E = h\nu$ converts to a mass m as shown in Figure.3. Here ν is the frequency of the radiation and h is the Planck's constant. This can happen at a minimum distance of wavelength of the photon i.e. λ. Therefore

$$\frac{\partial E}{\partial x} = -\frac{h\nu}{\lambda} \tag{21}$$

Negative sign shows energy decreases over the distance. Also momentum of the photon is mc and this process completes in time interval of $\frac{1}{\nu}$. Therefore

$$\frac{\partial p_x}{\partial t} = \frac{mc}{\frac{1}{\nu}} = mc\nu \tag{22}$$

We have from Equation (13(i))

$$\frac{\partial E}{\partial x} + \frac{\partial p_x}{\partial t} = 0 \tag{23}$$

Therefore from Equation (21) and (22),

$$-\frac{h\nu}{\lambda} + mc\nu = 0$$

$$\text{or } h\nu = mc\nu\lambda = mc^2 \text{ as } c = \nu\lambda \tag{24}$$

which is true by the Einstein Mass-Energy relation.

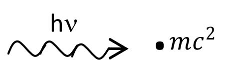

Figure 3: Photon of Energy $h\nu$ converts to mass m according to relation $h\nu = mc^2$.

D. Relation between Electromagnetic potentials

It is well known that in presence of electromagnetic fields the modifications in Energy E and Momentum **p** of an electron having charge –e can be represented as

$$E \to E - e\varphi, \qquad \mathbf{p} \to \mathbf{p} - \frac{e}{c}\mathbf{A} \tag{25}$$

where φ and A are the electric and magnetic potentials respectively. Let us assume the only variations in Energy are due to electromagnetic fields, then, in x - direction

$$\frac{\partial E}{\partial x} = -e\frac{\partial \varphi}{\partial x} \tag{26}$$

We have from Equation (13(i))

$$-\frac{\partial E}{\partial x} = \frac{\partial p_x}{\partial t}$$

We get, in presence of electromagnetic potentials,

$$e\frac{\partial \varphi}{\partial x} = \frac{\partial\left(p_x - \frac{eA_x}{c}\right)}{\partial t}$$

$$\frac{1}{e}\frac{\partial p_x}{\partial t} = \frac{\partial \varphi}{\partial x} + \frac{\partial A_x}{\partial (ct)} \tag{27}$$

Let $\mathbf{E} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$ and $\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$ are the Electric and Magnetic fields respectively. We know fields are related to the potentials by following relations

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \text{ and } \mathbf{B} = \nabla \times \mathbf{A} \tag{28}$$

Therefore
$$E_x = -\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial (ct)} \tag{29}$$

Combining Equation (27) and (29), we can write

$$\frac{1}{e}\frac{\partial p_x}{\partial t} = \frac{\partial \varphi}{\partial x} + \frac{\partial A_x}{\partial (ct)} = -E_x \tag{30}$$

Therefore Electric field in x-direction is the force experienced by a unit positive charge. Note here the charge is –e. This is also the known definition of E_x. Therefore

$$-E_x = \frac{1}{e}\frac{\partial p_x}{\partial t} = \frac{\partial \varphi}{\partial x} + \frac{\partial A_x}{\partial (ct)}$$

Multiplying numerator and denominator on right hand side by i, we get

$$-E_x = \frac{1}{e}\frac{\partial p_x}{\partial t} = \frac{\partial (i\varphi)}{i\,\partial x} + \frac{i\,\partial A_x}{\partial (ict)}$$

Multiplying throughout by i, we get

$$-iE_x = \frac{i}{e}\frac{\partial p_x}{\partial t} = \frac{\partial (i\varphi)}{\partial x} - \frac{\partial A_x}{\partial (ict)}$$

As we know

$$A^\mu = (A^0, A^1, A^2, A^3) = (i\varphi, A_x, A_y, A_z)$$ we can write

$$-iE_x = \frac{i}{e}\frac{\partial p_x}{\partial t} = \frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0}$$

Therefore

$$-iE_x = \frac{i}{e}\frac{\partial p_x}{\partial t} = \partial_1 A^0 - \partial_0 A^1 = -F^{01} \tag{31}$$

(Ignoring raising of indices on right hand side by the multiplication by the metric $g_{\mu\nu} = \delta^\mu_\nu$)

Here F^{01} is the t-x component of the Electro-magnetic Field Tensor. Similarly it can be shown that $F^{02} = iE_y$ and $F^{03} = iE_z$ using Equations (13)(ii) and (13)(iii) respectively.

Let us consider Equation (13)(iv)

$$\frac{\partial p_y}{\partial x} = \frac{\partial p_x}{\partial y}$$

We get, in presence of electromagnetic potentials,

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$$\frac{\partial\left(p_y-\frac{eA_y}{c}\right))}{\partial x}=\frac{\partial\left(p_x-\frac{eA_x}{c}\right)}{\partial y}$$

Rearranging terms, we get

$$\frac{\partial p_y}{\partial x}-\frac{\partial p_x}{\partial y}=\frac{e}{c}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

We know $\mathbf{B}=\nabla\times\mathbf{A}$, therefore

$$\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=B_z \text{ i.e. z component of } \mathbf{B}$$

Therefore above equation can be written as

$$\frac{\partial p_y}{\partial x}-\frac{\partial p_x}{\partial y}=\frac{e}{c}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)=\frac{e}{c}B_z \qquad (32)$$

Letm be the mass and $\mathbf{v}=v_x\hat{\mathbf{i}}+v_y\hat{\mathbf{j}}+v_z\hat{\mathbf{k}}$is the velocity of the electron, then left hand side of the equation becomes

$$\frac{\partial p_y}{\partial x}-\frac{\partial p_x}{\partial y}=m\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) \qquad (33)$$

But right hand side of Equation (33) is mass multiplied by the z-component of the curl of the velocity which again, as we know, is the twice of the z-component of the angular velocity of the electron say, ω. Mathematically

$$\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}=(\nabla\times v)_z=2\omega_z$$

Then from Equation (33)

$$\frac{\partial p_y}{\partial x}-\frac{\partial p_x}{\partial y}=2m\omega_z$$

Therefore substituting in Equation (32), we have

$$2m\omega_z=\frac{e}{c}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)=\frac{e}{c}B_z$$

or $$B_z=\frac{2mc}{e}\omega_z=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y} \qquad (34)$$

As $A^\mu=(A^0,A^1,A^2,A^3)=(i\varphi,A_x,A_y,A_z)$ we can write

$$\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=\frac{\partial A^2}{\partial x^1}-\frac{\partial A^1}{\partial x^2}$$
$$=\partial_1A^2-\partial_2A^1=F^{12}$$

Therefore

$$B_z=\frac{2mc}{e}\omega_z=\partial_1A^2-\partial_2A^1=F^{12} \qquad (35)$$

(Ignoring raising of indices on right hand side by the multiplication by the metric$g_{\mu\nu}=\delta^\mu_\nu$)

Therefore B_z is associated with the angular motion of the particle and is (1,2) component of the Electro-magnetic Field Tensor$F^{\mu\nu}$. Similarly it can be shown that$F^{31}=B_y=\frac{2mc}{e}\omega_y$and $F^{23}=B_x=\frac{2mc}{e}\omega_x$using Equations (13(v)) and (13(vi)) respectively.

We are known to the fact that if an electron with velocity $v$enters a perpendicular magnetic field**B**, it undergoes rotational motion with angular velocity ω. The Lorentz force provides for the necessary centripetal force. Therefore we can write mathematically

$$\frac{Bev}{c}=\frac{mv^2}{r} \text{ or } \quad B=\frac{mc}{e}\omega \qquad (36)$$

We are getting the same result but the appearance of a constant 2, however, requires an appropriate explanation. This we shall reserve for our future communication.Our present discussion, however, illustrates the fact that the magnetic field causes rotation and the angular velocity is proportional to the applied magnetic field.

Thus we have obtained thevarious components for the Electro-magnetic Field Tensor which can be arranged in the matrix form as below

$$F^{\mu\nu}=\begin{bmatrix}0 & iE_x & iE_y & iE_z\\ -iE_x & 0 & B_z & -B_y\\ -iE_y & -B_z & 0 & B_x\\ -iE_z & B_y & -B_x & 0\end{bmatrix} \qquad (37)$$

Transforming $t\rightarrow it,\varphi\rightarrow i\varphi,E_x\rightarrow iE_x$ and choosing metric$g^{\mu\nu}=\text{diag}\,(1,-1,-1,-1)$, we get Minkowski field tensor

$$F^{\mu\nu}=\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix} \qquad (38)$$

Therefore, we have obtained all the components of Electromagnetic tensor starting from our basic covariant equations (13).

These were few examples to illustrate the validity of $\boldsymbol{\partial}\,\boldsymbol{\Lambda}\,\mathbf{p}=\mathbf{0}$ as covariant form substitute for Newton's Second law.

IV. CONCLUSION

The spatial temporal variations of Energy-Momentum four-vector components are related through a covariant form of Newton's Second Law of motion. These relationships successfully explain the various pseudo-forces observed in nature and also confirmthe known relations between the electromagnetic potentials,fields and the field tensor.

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 1, pp: 1130-1135 www.ijaem.net ISSN: 2395-5252

ACKNOWLEDGEMENT

Author will like to thank Head, Department of Physics, Kumaun University, Nainital, Uttarakhand, India and Principal, R.H. Govt PG College, Kashipur (U.S. Nagar), India for their generous support in pursuing these studies.

REFERENCES

[1]. A. Einstein,A Brief Outline of the Development of the Theory of Relativity *Nature 106, 1921*, 782-784.
[2]. R.Adler,M.Bazin, , &M. Schiffer, *Introduction to General Relativity(Second Edition)* (McGraw-Hill, New York, 1975).
[3]. Bernard F. Schutz*A First Course in General Relativity (Second Edition)* (Cambridge University Press, New York, 2009).
[4]. Weblink: ww.cambridge.org/9780521887052

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International Journal of Advances in Engineering and Management
ISSN: 2395-5252

IJAEM

Volume: 02 Issue: 01 DOI: 10.35629/5252

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